数理解析研究所講究録 1341 巻 2003 年 52-76

Some Topics in N- Fractional Calculus

デカルト出版・西本勝之 (Katsuyuki Nishimoto)
Research Institute for Applied Mathematics
Descartes Press Co.

Abstract

In this article, the following three topics in N (Nishimoto's) - fractional calculus are reported, that is,

Part I. N- Fractional calculus operator N^{ν} (the set of them $\{N^{\nu}\}$ is an action group),

Part II. N- Fractional calculus of the function $\log(z-c)$ and Beta function,

Part III. Application of N-fractional calculus to the homogeneous Gauss equation and Kummer's 24 functions.

Part I. N- Fractional calculus operator N^{ν}

§ 0. Introduction (Definition of N Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_{-},D_{+}\}$, $C=\{C_{-},C_{+}\}$,

C_{-} be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_{+} be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_{-} be a domain surrounded by C_{-}, D_{+} be a domain surrounded by C_{+}.
(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_{\nu}=(f)_{\nu}={}_{C}(f)_{\nu}=\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^{-}), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_{\nu} \qquad (m\in Z^{+}), \tag{2}$$

where $-\pi\le\arg(\zeta-z)\le\pi$ for C_{-}, $0\le\arg(\zeta-z)\le 2\pi$ for C_{+},

$\zeta\ne z$, $z\in \boldsymbol{C}$, $\nu\in \boldsymbol{R}$, Γ ; Gamma function,

then $(f)_{\nu}$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z, of the function f, if $|(f)_{\nu}|<\infty$.

§ 1. The Set of N-Fractional Calculus Operator

[I] Definition of N- fractional calculus operator N^ν

We define N- fractional calculus operator (Nishimoto's Operator) N^ν as

$$N^\nu := \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z^-), \quad [\textit{Refer to} (1)] \tag{1}$$

with
$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in Z^+), \tag{2}$$

and define the binary operation $\circ = \times$ *as*

$$N^\beta \circ N^\alpha f = N^\beta \times N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in R). \tag{3}$$

Then we have

$$N^\nu f = N^\nu f(z) = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) (f(\zeta)) \qquad (\nu \notin Z^-), \tag{4}$$

$$= \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta \ , \tag{5}$$

$$= f_\nu(z) = f_\nu \ . \tag{6}$$

[I I] An Abelian product group

Following Theorem 1 is reported in JFC Vol. 4, Nov. (1994) [3]. However, it is shown again here for our convenience.

Theorem 1. *The set*

$$\{N^\nu\} = \{N^\nu \mid \nu \in R\} \tag{7}$$

is an Abelian product group (having continuous index $\nu \in R$, viz. $-\infty < \nu < \infty$ *) for the function*

$$f \in F = \{f;\ 0 \neq |f_\nu| < \infty, \nu \in R\}, \tag{8}$$

where $f = f(z)$ *and* $z \in C$.

(In the following $0 \neq f \in F$ and $\nu, \alpha, \beta, \gamma \in R$.)

Proof.

We have the following for the multiplication of N^ν.

We have ;

(i) $$N^\beta N^\alpha f = N^{\beta+\alpha} f = N^\gamma f \qquad (\gamma = \alpha + \beta). \tag{9}$$

Therefore, we obtain

$$N^\beta N^\alpha = N^{\beta+\alpha} = N^\gamma \in \{N^\nu\}. \qquad (\text{Closure}) \tag{10}$$

(ii) $$N^{\gamma}(N^{\beta}N^{\alpha})f = (N^{\gamma}N^{\beta})N^{\alpha}f = N^{\gamma+\beta+\alpha}f \quad . \tag{11}$$

Therefore, we obtain

$$N^{\gamma}(N^{\beta}N^{\alpha}) = (N^{\gamma}N^{\beta})N^{\alpha} \,. \quad \text{(Associative law)} \tag{12}$$

(iii) $$1 \cdot f = N^{0}f = f \quad . \tag{13}$$

Therefore, we obtain

$$1 = N^{0} \quad . \quad \text{(Existence of unit element)} \tag{14}$$

(iv) $$N^{-\nu}N^{\nu}f = N^{\nu}N^{-\nu}f = N^{0}f = f \quad . \tag{15}$$

Therefore, we obtain

$$N^{-\nu}N^{\nu} = N^{\nu}N^{-\nu} = N^{0} = 1. \quad \text{(Existence of inverse element)} \tag{16}$$

(v) $$N^{\beta}N^{\alpha}f = N^{\alpha}N^{\beta}f = N^{\alpha+\beta}f \quad . \tag{17}$$

Therefore, we obtain

$$N^{\beta}N^{\alpha} = N^{\alpha}N^{\beta} \quad . \quad \text{(Commutative law)} \tag{18}$$

Therefore, we have this Theorem 1 by (i) ~ (v).

Then we call the set $\{N^{\nu}\}$ as " Fractional calculus operator group " and denote this by " F.O.G. $\{N^{\nu}\}$ ".

Note 1. We have ([1] Vol.1, [2])

$$N^{\beta}N^{\alpha}f = (f_{\alpha})_{\beta} = \frac{\Gamma(\beta+1)}{2\pi i}\int_{C}\frac{f_{\alpha}(\eta)}{(\eta - z)^{\beta+1}}\,d\eta \tag{19}$$

$$= \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{(2\pi i)^{2}}\int_{C} f(\zeta)\,d\zeta \int_{C}\frac{d\eta}{(\zeta-\eta)^{\alpha+1}(\eta - z)^{\beta+1}} \tag{20}$$

$$= \frac{\Gamma(\alpha+\beta+1)}{2\pi i}\int_{C}\frac{f(\zeta)}{(\zeta - z)^{\alpha+\beta+1}}\,d\zeta \tag{21}$$

$$= f_{\alpha+\beta} = N^{\alpha+\beta}f \quad . \tag{22}$$

Note 2. Notice that letting $(N^{\nu})^{-1}$ be the inverse to N^{ν} we have

$$(N^{\nu})^{-1}N^{\nu} = N^{\nu}(N^{\nu})^{-1} = 1. \tag{23}$$

Then we obtain

$$(N^{\nu})^{-1} = N^{-\nu}, \tag{24}$$

from (24) and (16).

Therefore, we can see that

$$((N^{\nu})^{-1})^{-1} = (N^{-\nu})^{-1} = N^{\nu}, \tag{25}$$

from (24).

[I I I] Action group

We have the following definition for action group. ([17] pp. 40 - 42. & pp. 113 - 133.)

Definition 1. Let $G = \{g\}$ be a group, and $A = \{a\} \neq \phi$ be a set. When the map from $G \times A = \{(g, a) \mid g \in G, a \in A\}$ to $A = \{a \mid a \in A\}$ satisfies

$$\text{(i)} \quad g_1 \circ (g_2 a) = (g_1 \circ g_2) a \quad \text{for all} \quad g_1, g_2 \in G,\ a \in A\ , \tag{26}$$

$$\text{(ii)} \quad 1 \circ a = a \quad \text{for all} \quad a \in A\ , \tag{27}$$

we say "G is a group acting on a set A ".

Then we call G as " action group ".

Obeying this definition we have the following theorem.

Theorem 2. *The set $\{N^{\nu}\}$ is " an action group which has continuous index ν for the set F ".*

Proof. Letting $G = \{N^{\nu}\}$, $A = F$ and $a = f \in F$ in the above definition, we have

$$\text{(i)} \quad N^{\beta}(N^{\alpha} f) = (N^{\beta} N^{\alpha}) f \quad \text{for all} \quad N^{\beta}, N^{\alpha} \in \{N^{\nu}\},\ f \in F\ , \tag{28}$$

and

$$\text{(ii)} \quad N^{0} f = 1 \cdot f = f \quad \text{for all} \quad f \in F\ . \tag{29}$$

Therefore, we can see that the set $\{N^{\nu}\}$ of our fractional calculus operator N^{ν} is a group acting on a set F .

Theorem 3. *The set $\{N^{\nu}\}$ is " an Abelian product group acting on a set F , having continuous index $\nu \in R$.*

Proof. It is clear by Theorems 1. and 2.

Part II. N- Fractional calculus of the function $\log(z-c)$ and Beta function

Chapter 1. N- Fractional Calculus of the Power Functions and Logarithmic ones

§ 1. N- Fractional Calculus of Power Functions

Theorem 1. *We have*

$$\left((a-z)^{\beta}\right)_{\alpha} = \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(a-z)^{\beta-\alpha} \qquad \left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty\right) \tag{1}$$

where $z \in C$, $z \neq a$ *and* $\alpha(\in R)$, $\beta(\in R)$ *and* $a(\in C)$ *are constants.*

Proof. Obeying our definition of N- fractional calculus, we have

$$\left((a-z)^{q-1}\right)_{-p} = \frac{\Gamma(1-p)}{2\pi i}\int_{C} \frac{(a-\zeta)^{q-1}}{(\zeta-z)^{-p+1}}\,d\zeta \tag{2}$$

$$= \frac{\Gamma(1-p)}{2\pi i}\int_{-\infty+i\mathrm{Im}(z)}^{(z+)} \frac{a^{q-1}\{1-(\zeta/a)\}^{q-1}}{(\zeta-z)^{-p+1}}\,d\zeta \qquad \left(\begin{array}{c}\text{set } \zeta = a\xi,\\ a = \delta e^{i\phi};\ \delta,\phi \in R,\\ |\arg a| = |\phi| < \pi/2\end{array}\right) \tag{3}$$

$$= \frac{\Gamma(1-p)}{2\pi i}\,a^{p+q-1}\int_{-\infty+i\mathrm{Im}(t)}^{(t+)} \frac{(1-\xi)^{q-1}}{(\xi-t)^{-p+1}}\,d\xi \qquad \left(\begin{array}{c} z/a = t,\\ \xi - t = \eta\end{array}\right) \tag{4}$$

$$= \frac{\Gamma(1-p)}{2\pi i}\,a^{p+q-1}\int_{-\infty}^{(0+)} \eta^{p-1}(1-t-\eta)^{q-1}\,d\eta \qquad \left(\begin{array}{c} 1-t = s,\\ \eta = su\end{array}\right) \tag{5}$$

$$= \frac{\Gamma(1-p)}{2\pi i}\,(as)^{p+q-1}\int_{-\infty}^{(0+)} u^{p-1}(1-u)^{q-1}\,du \qquad \left(\begin{array}{c} |\psi| < \pi/2,\\ \psi = \arg s\end{array}\right) \tag{6}$$

$$= \frac{\Gamma(1-p)}{2\pi i}(a-z)^{p+q-1}\cdot 2i\sin\pi p \cdot B(1-p-q\,,\,p) \qquad ([\,2\,],\text{p.}20) \tag{7}$$

$$= \frac{\Gamma(1-q-p)}{\Gamma(1-q)}(a-z)^{p+q-1}\quad , \tag{8}$$

since we have

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin\pi p}\quad . \tag{9}$$

Therefore, setting

$$q-1 = \beta \qquad \text{and} \qquad -p = \alpha$$

we have (1) , under the conditions.

Corollary 1. *We have*

$$\left((z-a)^{\beta}\right)_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-a)^{\beta-\alpha} \qquad \left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty\right)\ . \tag{10}$$

Proof. It is clear by Theorem 1, since we have

$$e^{i\pi\beta}\left((z-a)^{\beta}\right)_{\alpha} = e^{i\pi(\beta-\alpha)}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-a)^{\beta-\alpha} \qquad \left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty\right) \tag{11}$$

from (1).

Corollary 2. *We have*

$$(1)_\alpha = 0 \qquad (|\Gamma(\alpha)| < \infty) \ . \tag{12}$$

Proof. We have

$$(1)_\alpha = \left(z^0\right)_\alpha = e^{-i\pi\alpha}\frac{\Gamma(\alpha-0)}{\Gamma(-0)} z^{0-\alpha} = 0 \qquad \left(|\Gamma(\alpha)| < \infty\right), \tag{13}$$

from (10).

§ 2. N- Fractional Calculus of Logarithmic Functions

Theorem 2. *We have*

(i) $$\left(\log(a-z)\right)_\alpha = -\Gamma(\alpha)(a-z)^{-\alpha} = -e^{-i\pi\alpha}\,\Gamma(\alpha)(z-a)^{-\alpha} \tag{14}$$

and

(i i) $$\left(\log(z-a)\right)_\alpha = \left(\log(a-z)\right)_\alpha \tag{15}$$

where $|\Gamma(\alpha)| < \infty$, $z \in C$, $z \neq a$, *and* $\alpha\,(\in R)$ *and* $a\,(\in C)$ *are constants.*

Proof of (i). We have

$$\left(\log(a-z)\right)_1 = -\,(a-z)^{-1} \qquad (z \neq a) \ . \tag{16}$$

Operate $N^{\alpha-1}$ to the both sides of (16), we have then

$$\left(\left(\log(a-z)\right)_1\right)_{\alpha-1} = -\,\left((a-z)^{-1}\right)_{\alpha-1} \ , \tag{17}$$

hence

$$\left(\log(a-z)\right)_\alpha = -\Gamma(\alpha)(a-z)^{-\alpha} \ ,$$

by our index law and by Theorem 1.

Proof of (i i). We have

$$\left(\log(a-z)\right)_\alpha = \left(\log e^{i\pi}(z-a)\right)_\alpha = \left(\log(z-a) + i\pi\right)_\alpha \tag{18}$$

$$= \left(\log(z-a)\right)_\alpha \tag{19}$$

since

$$\left(i\pi\right)_\alpha = 0 \quad \text{for} \quad |\Gamma(\alpha)| < \infty \ . \tag{20}$$

Therefore ,we have this theorem under the conditions.

Corollary 3. *We have*

$$(\log az)_\alpha = (\log z)_\alpha = -e^{-i\pi\alpha}\,\Gamma(\alpha)\,z^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty) \tag{21}$$

where $a \neq 0$.

Proof 1 . Since we have

$$(\log az)_\alpha = (\log z + \log a)_\alpha = (\log z)_\alpha + (\log a)_\alpha \ , \tag{22}$$

it is clear.

Proof 2. Using the relationship

$$\log az = \int_0^\infty \int_1^{az} e^{-ts}\,dt\,ds = \int_0^\infty \frac{e^{-s} - e^{-azs}}{s}\,ds \tag{23}$$

we obtain (21) by our definition of N- fractional calculus. ([1] Vol.1, pp 28 - 30.) ([2] pp. 50 - 51.)

Theorem 3. *We have*

(i) $$\left((a-z)^{-\alpha}\right)_{-\alpha} = -\frac{1}{\Gamma(\alpha)}\log(a-z) \tag{24}$$

and

(i i) $$\left((z-a)^{-\alpha}\right)_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-a) \tag{25}$$

where $|\Gamma(\alpha)|<\infty$, *and* $z \neq a$.

Proof of (i). We have

$$\left(\log(a-z)\right)_{\alpha} = -\Gamma(\alpha)(a-z)^{-\alpha} \qquad (|\Gamma(\alpha)|<\infty) \ . \tag{26}$$

Operate $N^{-\alpha}$ to the both sides of (26), we have then

$$\left(\left(\log(a-z)\right)_{\alpha}\right)_{-\alpha} = -\Gamma(\alpha)\left((a-z)^{-\alpha}\right)_{-\alpha} \quad , \tag{27}$$

hence we have

$$\log(a-z) = -\Gamma(\alpha)\left((a-z)^{-\alpha}\right)_{-\alpha} \tag{28}$$

by our index law.Therefore, we have (24) from (28) clearly, under the conditions.

Proof of (i i). We have

$$\left((a-z)^{-\alpha}\right)_{-\alpha} = e^{-i\pi\alpha}\left((z-a)^{-\alpha}\right)_{-\alpha} \ . \tag{29}$$

Therefore, we have (25) from (24), (29) and (15), under the conditions.

Theorem 4. *We have*

$$\left(\log(z-a)\right)_{-n} = \frac{(z-a)^n}{n!}\left\{\log(z-a) - \sum_{k=1}^{n}\frac{1}{k}\right\} \tag{30}$$

where $z \neq a$ *and* $n \in Z^+$.

Proof. We have

$$\left(\log(z-a)\right)_{-1} = \log(z-a)\cdot(z-a) - (z-a) \ . \tag{31}$$

Then operating N^{-m} ($m \in Z^+$) to the both sides of (31) we have

$$\left(\log(z-a)\right)_{-1-m} = \left(\log(z-a)\cdot(z-a)\right)_{-m} - (z-a)_{-m} \ . \tag{32}$$

Now we have

$$\left(\log(z-a)\cdot(z-a)\right)_{-m} = \sum_{k=0}^{1}\frac{\Gamma(1-m)}{k!\,\Gamma(1-m-k)}\left(\log(z-a)\right)_{-m-k}(z-a)_k \tag{33}$$

$$= \left(\log(z-a)\right)_{-m}(z-a) - m\left(\log(z-a)\right)_{-m-1} \tag{34}$$

since we have

$$(u\cdot v)_{\alpha} = \sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha)}{k!\,\Gamma(1+\alpha-k)}\,u_{\alpha-k}\,v_k \qquad ([\,1\,]\ \text{Vol.1}) \ , \tag{35}$$

$$(z-a)_{-m} = e^{i\pi m}\frac{\Gamma(-1-m)}{\Gamma(-1)}(z-a)^{1+m} = \frac{1}{(m+1)!}(z-a)^{1+m} \quad . \tag{36}$$

Therefore, substituting (36) and (34) into (32) we obtain

$$\left(\log(z-a)\right)_{-(1+m)} = \frac{1}{1+m}\left(\log(z-a)\right)_{-m}(z-a) - \frac{1}{(1+m)!\ (1+m)}(z-a)^{1+m} \ . \tag{37}$$

We have then, setting $m=0$,

$$\left(\log(z-a)\right)_{-1} = \log(z-a)\cdot(z-a)-(z-a)\ , \tag{31}$$

from (37), setting $m=1$ in (37) and using (31) we obtain

$$\left(\log(z-a)\right)_{-2} = \frac{1}{2}(z-a)^2\{\log(z-a)-(1+\tfrac{1}{2})\}\ . \tag{38}$$

Next setting $m=2$ in (37) ,and using (38) we obtain

$$\left(\log(z-a)\right)_{-3} = \frac{1}{3!}(z-a)^3\{\log(z-a)-(1+\tfrac{1}{2}+\tfrac{1}{3})\}\ , \tag{39}$$

and so on.

Therefore, we have this theorem from (39) for $z\neq a$ *and* $n\in Z^+$.

Note 1. S.-T. Tu and D-.K. Chyan derived

$$(z^\beta\cdot\log z)_\alpha = (z^\beta)_\alpha\{\log z+\psi(1+\beta)-\psi(1+\beta-\alpha)\} \quad (z\neq 0) \tag{42}$$

where

$$|\Gamma(\beta-\alpha)/\Gamma(-\alpha)|<\infty, \quad \mathrm{Re}\,(1+\beta)>0 \tag{43}$$

and ψ is the Psi function.

From (42) they got (30) having $a=0$. [6]

Chapter 2. N- Fractional Calculus of the Function $\log(z-c)$ and Beta Functions

§ 1. Some Theorems Associated with The Beta Functions for N- Fractional Calculus of Function $\log(z-c)$

In the following $\alpha,\beta,\gamma,\delta\in R$, $z\neq c$ and $B(\ ,\ )$ is Beta function.

Theorem 1. *We have*

$$\frac{\left(\log(z-c)\right)_{m+\alpha}\left(\log(z-c)\right)_{m+\beta}}{\left(\log(z-c)\right)_{n+\alpha+\beta}} = -e^{i\pi(n-2m)}\frac{[\alpha]_m[\beta]_m}{[\alpha+\beta]_n}B(\alpha,\beta)(z-c)^{n-2m} \tag{1}$$

where

$$m,n\in Z_0^+,\ (m+\alpha),(m+\beta),(n+\alpha+\beta)\notin Z_0^- \quad \textit{and}$$

$$[\lambda]_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad with \quad [\lambda]_0 = 1 .$$

Proof. We have

$$\left(\log(z-c)\right)_{m+\alpha} = -e^{-i\pi(m+\alpha)}\Gamma(m+\alpha)(z-c)^{-(m+\alpha)} \qquad \left(\left|\Gamma(m+\alpha)\right| < \infty\right), \qquad (2)$$

$$\left(\log(z-c)\right)_{m+\beta} = -e^{-i\pi(m+\beta)}\Gamma(m+\beta)(z-c)^{-(m+\beta)} \qquad \left(\left|\Gamma(m+\beta)\right| < \infty\right), \qquad (3)$$

and

$$\left(\log(z-c)\right)_{n+\alpha+\beta} = -e^{-i\pi(n+\alpha+\beta)}\Gamma(n+\alpha+\beta)(z-c)^{-(n+\alpha+\beta)} \quad \left(\left|\Gamma(n+\alpha+\beta)\right| < \infty\right), \qquad (4)$$

Therefore, making $(2)\times(3)/(4)$, we have

$$\text{LHS of } (1) = -e^{i\pi(n-2m)}\frac{[\alpha]_m[\beta]_m}{[\alpha+\beta]_n}\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(z-c)^{n-2m} \qquad (5)$$

$$= -e^{i\pi(n-2m)}\frac{[\alpha]_m[\beta]_m}{[\alpha+\beta]_n}B(\alpha,\beta)(z-c)^{n-2m} , \qquad (1)$$

under the conditions, since

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta). \qquad (6)$$

Corollary 1. *We have*

(i)
$$\frac{\left(\log(z-c)\right)_\alpha\left(\log(z-c)\right)_\beta}{\left(\log(z-c)\right)_{\alpha+\beta}} = -B(\alpha,\beta) \qquad (\text{Theorem A}) \qquad (7)$$

where

$$\alpha, \beta, (\alpha+\beta) \notin Z_0^- .$$

(ii)
$$\frac{\left(\log(z-c)\right)_{1+\alpha}\left(\log(z-c)\right)_{1+\beta}}{\left(\log(z-c)\right)_{1+\alpha+\beta}} = \frac{\alpha\beta}{\alpha+\beta}B(\alpha,\beta)(z-c)^{-1} \qquad (8)$$

where

$$(1+\alpha), (1+\beta), (1+\alpha+\beta) \notin Z_0^- .$$

(iii)
$$\frac{\left(\log(z-c)\right)_{1+\alpha}\left(\log(z-c)\right)_{1+\beta}}{\left(\log(z-c)\right)_{2+\alpha+\beta}} = -\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}B(\alpha,\beta) \qquad (9)$$

where

$$(1+\alpha), (1+\beta), (2+\alpha+\beta) \notin Z_0^- .$$

(iv) $$\frac{\left[(\log(z-c))_{m+\alpha}\right]^2}{(\log(z-c))_{m+2\alpha}} = (-1)^{m+1}\frac{([\alpha]_m)^2}{[2\alpha]_m} B(\alpha,\beta)(z-c)^{-m} \qquad (10)$$

where

$$m \in Z_0^+ , \quad (m+\alpha), (m+2\alpha) \notin Z_0^- \ .$$

Proof of (i). Set $m = n = 0$ in (1).

Proof of (i i). Set $m = n = 1$ in (1).

Proof of (i i i). Set $m = 1$, $n = 2$ in (1).

Proof of (i v). Set $\alpha = \beta$, $m = n$ in (1).

Theorem 2. *We have*

$$\frac{\left(\log(\log(z-c))_\gamma\right)_\alpha \left(\log(\log(z-c))_\gamma\right)_\beta}{\left(\log(\log(z-c))_\gamma\right)_{\alpha+\beta}} = \gamma\ B(\alpha,\beta) \qquad (11)$$

where

$$\gamma, \alpha, \beta, (\alpha+\beta) \notin Z_0^- \ .$$

Proof. We have

$$\left(\log(z-c)\right)_\gamma = -e^{-i\pi\gamma}\Gamma(\gamma)(z-c)^{-\gamma} \qquad \left(|\Gamma(\gamma)| < \infty\right), \qquad (12)$$

$$\left(\log(z-c)\right)_{\gamma+1} = -e^{-i\pi(\gamma+1)}\Gamma(\gamma+1)(z-c)^{-(\gamma+1)} \qquad \left(|\Gamma(\gamma+1)| < \infty\right), \qquad (13)$$

hence we have

$$\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma} = -\gamma\,(z-c)^{-1} \qquad (14)$$

from (12) and (13).

Then operating N^{-1} to the both sides fo (14) we otain

$$\left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_{-1} = -\gamma\left((z-c)^{-1}\right)_{-1} , \qquad (15)$$

namely

$$\log(\log(z-c))_\gamma = -\gamma\log(z-c). \qquad (16)$$

Next operate N^α to the both sides fo (16), we have then

$$\left(\log(\log(z-c))_\gamma\right)_\alpha = -\gamma\left(\log(z-c)\right)_\alpha \qquad (17)$$

$$= \gamma\, e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty). \qquad (18)$$

In the same way we obtain

$$\left(\log(\log(z-c))_\gamma\right)_\beta = \gamma e^{-i\pi\beta}\Gamma(\beta)(z-c)^{-\beta} \qquad (|\Gamma(\beta)|<\infty) . \tag{19}$$

and

$$\left(\log(\log(z-c))_\gamma\right)_{\alpha+\beta} = \gamma e^{-i\pi(\alpha+\beta)}\Gamma(\alpha+\beta)(z-c)^{-(\alpha+\beta)} \qquad (|\Gamma(\alpha+\beta)|<\infty) . \tag{20}$$

Therefore, making (18) × (19) /(20), we obtain

$$\text{LHS of (11)} = \gamma \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \gamma \cdot B(\alpha,\beta) , \tag{11}$$

under the conditions.

Corollary 2. *We have*

$$\frac{\left[\left(\log(\log(z-c))_\gamma\right)_\alpha\right]^2}{\left(\log(\log(z-c))_\gamma\right)_{2\alpha}} = \gamma \cdot B(\alpha,\alpha) \tag{21}$$

where

$$\gamma , \alpha , \ 2\alpha \notin \mathbf{Z}_0^- \ .$$

Proof. Set $\alpha = \beta$ in (11).

Theorem 3. *Let*

$$S = S(z) = \log(\log(\log(z-c))_\gamma)_\delta \ . \tag{22}$$

We have then

$$\frac{S_\alpha S_\beta}{S_{\alpha+\beta}} = \delta \cdot B(\alpha,\beta) \tag{23}$$

where

$$\delta , \gamma , \alpha , \beta , (\alpha+\beta) \notin \mathbf{Z}_0^- \ .$$

Proof. We have

$$\left(\log(\log(z-c))_\gamma\right)_\delta = -\gamma(\log(z-c))_\delta \qquad (|\Gamma(\gamma)|<\infty) \tag{24}$$

$$= \gamma e^{-i\pi\delta}\Gamma(\delta)(z-c)^{-\delta} \qquad (|\Gamma(\delta)|<\infty) . \tag{25}$$

and

$$\left(\log(\log(z-c))_\gamma\right)_{\delta+1} = \gamma e^{-i\pi(\delta+1)}\Gamma(\delta+1)(z-c)^{-\delta-1} \qquad (|\Gamma(\delta+1)|<\infty) . \tag{26}$$

from (16), respectively.

Then, making (26) /(25), we obtain

$$\frac{\left(\log(\log(z-c))_\gamma\right)_{\delta+1}}{\left(\log(\log(z-c))_\gamma\right)_\delta} = -\delta \cdot (z-c)^{-1} \ . \tag{27}$$

Then operating N^{-1} to the both sides fo (27) we otain

$$\left(\frac{\left(\log(\log(z-c))_\gamma\right)_{\delta+1}}{\left(\log(\log(z-c))_\gamma\right)_\delta}\right)_{-1} = -\delta\cdot\left((z-c)^{-1}\right)_{-1} , \qquad (28)$$

that is,

$$S = \log(\log(\log(z-c))_\gamma)_\delta = -\delta\cdot\log(z-c). \qquad (29)$$

Next operate N^α to the both sides of (29), we have then

$$S_\alpha = \delta\cdot e^{-i\pi\alpha}\,\Gamma(\alpha)(z-c)^{-\alpha} \quad (\,|\,\Gamma(\alpha)\,|<\infty)\,. \qquad (30)$$

In the same way we obtain

$$S_\beta = \delta\cdot e^{-i\pi\beta}\,\Gamma(\beta)(z-c)^{-\beta} \quad (\,|\Gamma(\beta)|<\infty)\,. \qquad (31)$$

and

$$S_{\alpha+\beta} = \delta\cdot e^{-i\pi(\alpha+\beta)}\,\Gamma(\alpha+\beta)(z-c)^{-\alpha-\beta} \quad (\,|\Gamma(\alpha+\beta)|<\infty)\,. \qquad (32)$$

Therefore, making (30) $\times$(31)/(32), we obtain

$$\frac{S_\alpha S_\beta}{S_{\alpha+\beta}} = \delta\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \delta\cdot B(\alpha,\beta) \qquad (23)$$

under the conditions.

Corollary 3. *We have*

$$\frac{(S_\alpha)^2}{S_{2\alpha}} = \delta\cdot B(\alpha,\alpha) \qquad (33)$$

where

$$\delta\,,\gamma\,,\alpha\,,\ 2\alpha\notin \boldsymbol{Z}_0^-\ .$$

Proof. Set $\alpha=\beta$ in (23).

Theorem 4. *Let*

$$T = T(z) = \frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\,. \qquad (34)$$

We have then

$$\frac{T_\alpha T_\beta}{T_{\alpha+\beta}} = -\frac{\alpha\beta\gamma}{\alpha+\beta}\cdot B(\alpha,\beta)(z-c)^{-1}\ , \qquad (35)$$

where

$$\gamma\,,\alpha\,,\beta\,,(\alpha+\beta)\notin \boldsymbol{Z}_0^-\ .$$

Proof. Operating N^α to the both sides of (34) we have

$$T_\alpha = \left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_\alpha = -\gamma\left((z-c)^{-1}\right)_\alpha \quad (\,|\Gamma(\gamma)|<\infty)\,. \qquad (36)$$

$$= -\gamma\,\alpha e^{-i\pi\alpha}\,\Gamma(\alpha)(z-c)^{-1-\alpha} \quad (\,|\Gamma(\alpha)|<\infty) \,. \tag{37}$$

In the same way, we obtain

$$T_\beta = -\gamma\,\beta e^{-i\pi\beta}\,\Gamma(\beta)(z-c)^{-1-\beta} \quad (\,|\Gamma(\beta)|<\infty)\,, \tag{38}$$

and

$$T_{\alpha+\beta} = -\gamma\,(\alpha+\beta)e^{-i\pi(\alpha+\beta)}\,\Gamma(\alpha+\beta)\,(z-c)^{-1-\alpha-\beta} \quad (\,|\Gamma(\alpha+\beta)|<\infty) \tag{39}$$

from (34).

Therefore, we have

$$\frac{T_\alpha T_\beta}{T_{\alpha+\beta}} = -\frac{\alpha\beta\gamma}{\alpha+\beta}\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(z-c)^{-1}\,, \tag{40}$$

from (37), (38) and (39).

We have then (35) from (40) clearly, under the conditions.

Corollary 4. *We have*

$$\frac{(T_\alpha)^2}{T_{2\alpha}} = -\frac{\alpha\gamma}{2}\cdot B(\alpha,\alpha)(z-c)^{-1} \tag{41}$$

where

$$\gamma\,,\alpha\,,\ 2\alpha \notin Z_0^- \ .$$

Proof. Set $\alpha=\beta$ in (35).

§ 2. Some examples of the Theorems

(I) Example of Theorem 1

(i) $$\frac{(\log(z-c))_{1/2}(\log(z-c))_{3/2}}{(\log(z-c))_{(1/2)+(3/2)}} = -B(1/2,3/2) = -\frac{\pi}{2}\ . \tag{1}$$

(ii) $$\frac{(\log(z-c))_2(\log(z-c))_3}{(\log(z-c))_4} = \frac{1\cdot 2}{3}B(1,2)(z-c)^{-1} = \frac{1}{3}(z-c)^{-1}\ . \tag{2}$$

(iii) $$\frac{(\log(z-c))_{1+(1/2)}(\log(z-c))_{1+(1/3)}}{(\log(z-c))_{1+(1/2)+(1/3)}} = \frac{(1/2)(1/3)}{((1/2)+(1/3))}B(1/2,3/2)\,(z-c)^{-1}$$

$$= \frac{1}{5}B(1/2,1/3)(z-c)^{-1}\ . \tag{3}$$

(iv) $$\frac{[(\log(z-c))_{1+(1/2)}]^2}{(\log(z-c))_2} = \left(\frac{1}{2}\right)^2 B(1/2,1/2)(z-c)^{-1} = \frac{\pi}{4}(z-c)^{-1}\ . \tag{4}$$

(II) **Example of Theorem 2**

(i)
$$\frac{\left(\log(\log(z-c))_\gamma\right)_{1/2}\left(\log(\log(z-c))_\gamma\right)_{3/2}}{\left(\log(\log(z-c))_\gamma\right)_2} = \gamma\cdot B(1/2,3/2) = \frac{\gamma\pi}{2} \ . \qquad (5)$$

(ii)
$$\frac{[\left(\log(\log(z-c))_\gamma\right)_{1/2}]^2}{\left(\log(\log(z-c))_\gamma\right)_1} = \gamma\cdot B(1/2,1/2) = \gamma\cdot\pi \ . \qquad (6)$$

(III) **Example of Theorem 3**

(i)
$$\frac{S_{1/2}S_{3/2}}{S_2} = \frac{\left(\log(\log(\log(z-c))_\gamma)_\delta\right)_{1/2}\left(\log(\log(\log(z-c))_\gamma)_\delta\right)_{3/2}}{\left(\log(\log(\log(z-c))_\gamma)_\delta\right)_2}$$

$$= \delta\cdot B(1/2,3/2) = \frac{\delta\pi}{2} \ . \qquad (7)$$

(ii)
$$\frac{(S_{1/2})^2}{S_1} = \frac{[\left(\log(\log(\log(z-c))_\gamma)_\delta\right)_{1/2}]^2}{\left(\log(\log(\log(z-c))_\gamma)_\delta\right)_1} = \delta\cdot B(1/2,1/2) = \delta\cdot\pi \ . \qquad (8)$$

(IV) **Example of Theorem 4**

(i)
$$\frac{T_{1/2}T_{3/2}}{T_2} = \left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_{1/2}\cdot\left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_{3/2} \Big/ \left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_2$$

$$= -\frac{3}{8}\gamma\cdot B(1/2,3/2)(z-c)^{-1} = -\frac{3}{16}\gamma\,\pi\,(z-c)^{-1} \ . \qquad (9)$$

(ii)
$$\frac{(T_{1/2})^2}{T_1} = \left[\left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_{1/2}\right]^2 \Big/ \left(\frac{(\log(z-c))_{\gamma+1}}{(\log(z-c))_\gamma}\right)_1$$

$$= -\frac{\gamma}{4}\cdot B(1/2,1/2)(z-c)^{-1} = -\frac{1}{4}\gamma\pi\cdot(z-c)^{-1} \ . \qquad (10)$$

Part III. Application of N-fractional calculus to the homogeneous Gauss equation and Kummer's 24 functions.

Chapter 1. N-fractional calculus operator N^ν method to homogeneous Gauss equation

§1. N^ν method to the homogeneous Gauss equation

By our fractional calculus operator N^ν method we obtain the following solutions which contain the N-fractional calculus.

Theorem 1. *Let* $\varphi \in \wp^{\circ} = \{\varphi \mid 0 \neq |\varphi_\nu| < \infty,\ \nu \in R\}$, *then the homogeneous Gauss equation*

$$L[\varphi, z; \alpha, \beta, \gamma]$$
$$= \varphi_2 \cdot (z^2 - z) + \varphi_1 \cdot \{z(\alpha + \beta + 1) - \gamma\} + \varphi \cdot \alpha\beta = 0 \qquad (z \neq 0, 1) \tag{0}$$

has solutions of the form

(Group I);

$$\varphi = K\left(z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1}\right)_{\alpha-1} \equiv \varphi_{(1)}, \quad (\textit{denote}) \tag{1}$$
$$\varphi = K\left(z^{\beta-\gamma} \cdot (z-1)^{\gamma-\alpha-1}\right)_{\beta-1} \equiv \varphi_{(2)}, \tag{2}$$
$$\varphi = K\left((z-1)^{\gamma-\beta-1} \cdot z^{\alpha-\gamma}\right)_{\alpha-1} \equiv \varphi_{(3)}, \tag{3}$$
$$\varphi = K\left((z-1)^{\gamma-\alpha-1} \cdot z^{\beta-\gamma}\right)_{\beta-1} \equiv \varphi_{(4)}, \tag{4}$$

(Group II);

$$\varphi = Kz^{1-\gamma}\left(z^{\alpha-1} \cdot (z-1)^{-\beta}\right)_{\alpha-\gamma} \equiv \varphi_{(5)}, \tag{5}$$
$$\varphi = Kz^{1-\gamma}\left(z^{\beta-1} \cdot (z-1)^{-\alpha}\right)_{\beta-\gamma} \equiv \varphi_{(6)}, \tag{6}$$
$$\varphi = Kz^{1-\gamma}\left((z-1)^{-\beta} \cdot z^{\alpha-1}\right)_{\alpha-\gamma} \equiv \varphi_{(7)}, \tag{7}$$
$$\varphi = Kz^{1-\gamma}\left((z-1)^{-\alpha} \cdot z^{\beta-1}\right)_{\beta-\gamma} \equiv \varphi_{(8)}, \tag{8}$$

(Group III);

$$\varphi = K(z-1)^{\gamma-\alpha-\beta}\left(z^{-\alpha} \cdot (z-1)^{\beta-1}\right)_{\gamma-\alpha-1} \equiv \varphi_{(9)}, \tag{9}$$
$$\varphi = K(z-1)^{\gamma-\alpha-\beta}\left(z^{-\beta} \cdot (z-1)^{\alpha-1}\right)_{\gamma-\beta-1} \equiv \varphi_{(10)}, \tag{10}$$
$$\varphi = K(z-1)^{\gamma-\alpha-\beta}\left((z-1)^{\beta-1} \cdot z^{-\alpha}\right)_{\gamma-\alpha-1} \equiv \varphi_{(11)}, \tag{11}$$
$$\varphi = K(z-1)^{\gamma-\alpha-\beta}\left((z-1)^{\alpha-1} \cdot z^{-\beta}\right)_{\gamma-\beta-1} \equiv \varphi_{(12)}, \tag{12}$$

where $\varphi_k = d^k\varphi/dz^k\ (k = 0, 1, 2)$, $\varphi_0 = \varphi = \varphi(z)$, $z \in C$, *and* K *is an arbitrary constant,* α, β *and* γ *are given constants.*

Proof of Group I;

Operate N-fractional calculus operator N^ν directly to the both sides of (0), we have then

$$N^\nu\{L[\varphi, z; \alpha, \beta, \gamma]\}$$
$$= \varphi_{2+\nu} \cdot (z^2 - z) + \varphi_{1+\nu} \cdot \{z(2\nu + \alpha + \beta + 1) - \nu - \gamma\}$$
$$+ \varphi_\nu \cdot \{\nu^2 + \nu(\alpha + \beta) + \alpha\beta\} = 0 \qquad (z \neq 0, 1) \tag{13}$$

since

$$N^\nu(\varphi_m \cdot z^n) = (\varphi_m \cdot z^n)_\nu = \sum_{k=0}^{n} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-k)\Gamma(k+1)} (\varphi_m)_{\nu-k}(z^n)_k \tag{14}$$

where $n \in Z_0^+\ (= Z^+ \cup \{0\})$.

Choose ν such that $\qquad \nu^2 + \nu(\alpha + \beta) + \alpha\beta = 0, \qquad (15)$

we have then $\qquad \nu = -\alpha$ *and* $-\beta$. $\qquad (16)$

Substitute $\nu = -\alpha$ into (13), yield

$$\varphi_{2-\alpha}\cdot(z^2-z)+\varphi_{1-\alpha}\cdot\{z(-\alpha+\beta+1)+\alpha-\gamma\}=0\,. \tag{17}$$

Therefore, setting

$$\varphi_{1-\alpha}=u=u(z) \qquad (\varphi=u_{\alpha-1}) \tag{18}$$

we have

$$u_1+u\cdot\frac{z(-\alpha+\beta+1)+\alpha-\gamma}{z^2-z}=0 \qquad (z\neq 0,1) \tag{19}$$

from (17). The solution of this differential equation is given by

$$u=Kz^{\alpha-\gamma}(z-1)^{\gamma-\beta-1}\,. \tag{20}$$

Thus we obtain

$$\varphi=K(z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1})_{\alpha-1}\equiv\varphi_{(1)} \qquad (denote) \quad (z\neq 0,1) \quad (1)$$

from (20) and (18). Where K is an arbitrary constant.

Inversely, the function given by (20) satisfies (19) clearly. Hence (1) satisfies equation (17). Therefore, the function (1) satisfies equation (0).

For $\nu=-\beta$, in the same way (or merely by the change of α and β in (1), because the equation (0) is symmetry for α and β) we obtain other solution

$$\varphi=K\left(z^{\beta-\gamma}\cdot(z-1)^{\gamma-\alpha-1}\right)_{\beta-1}\equiv\varphi_{(2)} \qquad (z\neq 0,1) \qquad (2)$$

which is different from (1), if $\alpha\neq\beta$.

Moreover, changing the order $z^{\alpha-\gamma}$ and $(z-1)^{\gamma-\beta-1}$ in (1) we have other solution ([6] Vol. 1 & [7])

$$\varphi=K\left((z-1)^{\gamma-\beta-1}\cdot z^{\alpha-\gamma}\right)_{\alpha-1}\equiv\varphi_{(3)} \qquad (z\neq 0,1) \qquad (3)$$

different from (1) when $(\alpha-1)\notin Z_0^+$. In the same way we have other solution

$$\varphi=K\left((z-1)^{\gamma-\alpha-1}\cdot z^{\beta-\gamma}\right)_{\beta-1}\equiv\varphi_{(4)} \qquad (z\neq 0,1) \qquad (4)$$

from (2), which is different from (2) when $(\beta-1)\notin Z_0^+$.

Proof of Group II;

Set
$$\varphi=z^\lambda\phi, \qquad \phi=\phi(z) \qquad (z\neq 0,1) \tag{21}$$

(Hence $\varphi_1=\lambda z^{\lambda-1}\phi+z^\lambda\phi_1$ and $\varphi_2=\lambda(\lambda-1)z^{\lambda-2}\phi+2\lambda z^{\lambda-1}\phi_1+z^\lambda\phi_2$).

Substitute (21) into (0), we have then

$$\begin{aligned}&\phi_2\cdot(z^2-z)+\phi_1\cdot\left\{z(\alpha+\beta+1+2\lambda)-2\lambda-\gamma\right\}\\&\quad+\phi\cdot\left\{\lambda(\lambda-1)+\lambda(\alpha+\beta+1)+\alpha\beta-z^{-1}\lambda(\lambda-1+\gamma)\right\}=0\end{aligned} \tag{22}$$

where $\phi_k=d^k\phi/dz^k$ $(k=0,1,2)$ and $\phi_0=\phi$.

Here, we choose λ such that

$$\lambda(\lambda-1+\gamma)=0 \tag{23}$$

that is,
$$\lambda=0,\ \ 1-\gamma\,. \tag{24}$$

(i) In the case $\lambda=0$

In this case we have the same results as Group I.

(ii) In the case $\lambda=1-\gamma$

Substituting $\lambda=1-\gamma$ into (22) we have

$$\phi_2\cdot(z^2-z)+\phi_1\cdot\left\{z(\alpha+\beta-2\gamma+3)+\gamma-2\right\}+\phi\cdot\left\{(1-\gamma)+\alpha\right\}\left\{(1-\gamma)+\beta\right\}=0\,. \tag{25}$$

Next, operate N^ν to the both sides of (25), we have then

$$\begin{aligned}&\phi_{2+\nu}\cdot(z^2-z)+\phi_{1+\nu}\cdot\left\{z(\alpha+\beta-2\gamma+3+2\nu)+\gamma-2-\nu\right\}\\&\quad+\phi_\nu\cdot\left\{\nu^2+\nu(\alpha+\beta-2\gamma+2)+(1-\gamma)(\alpha+\beta+1-\gamma)+\alpha\beta\right\}=0\,.\end{aligned} \tag{26}$$

Here we choose ν such that

$$\nu^2+\nu(\alpha+\beta-2\gamma+2)+(1-\gamma)(\alpha+\beta+1-\gamma)+\alpha\beta=0\,, \qquad (27)$$

that is,

$$\nu=\gamma-\alpha-1 \quad (28), \qquad \text{and} \qquad \nu=\gamma-\beta-1 \quad (29).$$

1) For the case of (28);

Substituting (28) into (26), we have

$$\phi_{1+\gamma-\alpha}\cdot(z^2-z)+\phi_{\gamma-\alpha}\cdot\{z(\beta-\alpha+1)+\alpha-1\}=0\,. \qquad (30)$$

Set

$$\phi_{\gamma-\alpha}=u=u(z) \qquad (\phi=u_{\alpha-\gamma})\,, \qquad (31)$$

we have then

$$u_1\cdot(z^2-z)+u\cdot\{z(\beta-\alpha+1)+\alpha-1\}=0 \qquad (32)$$

from (30). The solution to this differential equation is given by

$$u=Kz^{\alpha-1}(z-1)^{-\beta} \qquad (z\neq 0,1) \qquad (33)$$

where K is an arbitrary constant.

Therefore, we obtain

$$\phi=K\left(z^{\alpha-1}\cdot(z-1)^{-\beta}\right)_{\alpha-\gamma} \qquad (z\neq 0,1) \qquad (34)$$

from (33) and (31), hence we have

$$\varphi=Kz^{1-\gamma}\left(z^{\alpha-1}\cdot(z-1)^{-\beta}\right)_{\alpha-\gamma}\equiv\varphi_{(5)} \qquad (z\neq 0,1) \qquad (5)$$

from (34) and (21).

Inversely, (33) satisfies (32), then (34) satisfies (30) clearly. Therefore, (5) satisfies (0), since we have (21).

2) For the case of (29);

In the same way as 1) (or merely by the change α and β in (5)) we obtain

$$\varphi=Kz^{1-\gamma}\left(z^{\beta-1}\cdot(z-1)^{-\alpha}\right)_{\beta-\gamma}\equiv\varphi_{(6)} \qquad (z\neq 0,1) \qquad (6)$$

as the solutions to the equation (0), which is different from (5) if $\alpha\neq\beta$.

Moreover, changing the order $z^{\alpha-1}$ and $(z-1)^{-\beta}$ in (5) we have other solution

$$\varphi=Kz^{1-\gamma}\left((z-1)^{-\beta}\cdot z^{\alpha-1}\right)_{\alpha-\gamma}\equiv\varphi_{(7)} \qquad (z\neq 0,1) \qquad (7)$$

different from (5) when $(\alpha-\gamma)\notin Z_0^+$. In the same way we have other solution

$$\varphi=Kz^{1-\gamma}\left((z-1)^{-\alpha}\cdot z^{\beta-1}\right)_{\beta-\gamma}\equiv\varphi_{(8)} \qquad (z\neq 0,1) \qquad (8)$$

from (6), which is different from (6) when $(\beta-\gamma)\notin Z_0^+$.

Proof of Group III;

Set $\qquad \varphi=(z-1)^{\lambda}\phi, \qquad \phi=\phi(z) \quad (z\neq 0,1) \qquad (35)$

and substitute (35) into (0), we have then

$$\phi_2\cdot(z^2-z)+\phi_1\cdot\{z(\alpha+\beta+1+2\lambda)-\gamma\}$$
$$+\phi\cdot\{(z-1)^{-1}(\lambda^2+\lambda\alpha+\lambda\beta-\lambda\gamma)+(\lambda+\alpha)(\lambda+\beta)\}=0\,. \qquad (36)$$

Here, we choose λ such that

$$\lambda(\lambda+\alpha+\beta-\gamma)=0\,, \qquad (37)$$

that is,

$$\lambda=0,\ \gamma-\alpha-\beta\,. \qquad (38)$$

(i) In the case $\lambda=0$

In this case we have the same results as Group I.

(ii) In the case $\lambda = \gamma - \alpha - \beta$

In this case, substituting $\lambda = \gamma - \alpha - \beta$ into (36) we have

$$\phi_2 \cdot (z^2 - z) + \phi_1 \cdot \{z(2\gamma - \alpha - \beta + 1) - \gamma\} + \phi \cdot (\gamma - \alpha)(\gamma - \beta) = 0 . \tag{39}$$

Next operate N^ν to the both sides of (39), we ahve then

$$\phi_{2+\nu} \cdot (z^2 - z) + \phi_{1+\nu} \cdot \{z(2\nu + 2\gamma - \alpha - \beta + 1) - \nu - \gamma\}$$
$$+ \phi_\nu \cdot \{\nu^2 + \nu(2\gamma - \alpha - \beta) + (\gamma - \alpha)(\gamma - \beta)\} = 0 . \tag{40}$$

Here, we choose ν such that

$$\nu^2 + \nu(2\gamma - \alpha - \beta) + (\gamma - \alpha)(\gamma - \beta) = 0 , \tag{41}$$

that is,

$$\nu = \alpha - \gamma \quad (42), \qquad \text{and} \qquad \nu = \beta - \gamma . \tag{43}$$

1) For the case of (42);

Substituting (42) into (40), we have

$$\phi_{2+\alpha-\gamma} \cdot (z^2 - z) + \phi_{1+\alpha-\gamma} \cdot \{z(\alpha - \beta + 1) - \alpha\} = 0 . \tag{44}$$

Next set

$$\phi_{1+\alpha-\gamma} = w = w(z) \qquad (\phi = w_{\gamma-\alpha-1}), \tag{45}$$

we have then

$$w_1 \cdot (z^2 - z) + w \cdot \{z(\alpha - \beta + 1) - \alpha\} = 0 \tag{46}$$

from (44). The solution to this equation is given by

$$w = K z^{-\alpha} (z - 1)^{\beta - 1} \tag{47}$$

where K is an arbitrary constant.

Hence we have

$$\phi = K\left(z^{-\alpha}(z - 1)^{\beta - 1}\right)_{\gamma-\alpha-1} \tag{48}$$

from (47) and (45). Therefore we have

$$\varphi = K(z - 1)^{\gamma-\alpha-\beta}\left(z^{-\alpha} \cdot (z - 1)^{\beta-1}\right)_{\gamma-\alpha-1} \equiv \varphi_{(9)} \qquad (denote) \quad (z \neq 0, 1) \tag{9}$$

from (48) and (35) which has $\lambda = \gamma - \alpha - \beta$.

Inversely, (47) satisfies (46), then (48) satisfies (44) clearly. Therefore, (9) satisfies (0), since we have (35).

2) For the case of (43);

In the same way (or merely by the change α and β in (9)) we obtain

$$\varphi = K(z - 1)^{\gamma-\alpha-\beta}\left(z^{-\beta}(z - 1)^{\alpha-1}\right)_{\gamma-\beta-1} \equiv \varphi_{(10)} . \quad (z \neq 0, 1) \tag{10}$$

Moreover, changing the order $z^{-\alpha}$ and $(z-1)^{\beta-1}$ in (9) we have other solution

$$\varphi = K(z - 1)^{\gamma-\alpha-\beta}\left((z - 1)^{\beta-1} z^{-\alpha}\right)_{\gamma-\alpha-1} \equiv \varphi_{(11)} \quad (z \neq 0, 1) \tag{11}$$

which is different from (9) when $(\gamma - \alpha - 1) \notin Z_0^+$.

In the same way we have other solution

$$\varphi = K(z - 1)^{\gamma-\alpha-\beta}\left((z - 1)^{\alpha-1} z^{-\beta}\right)_{\gamma-\beta-1} \equiv \varphi_{(12)} \quad (z \neq 0, 1) \tag{12}$$

from (10), which is different from (10) when $(\gamma - \beta - 1) \notin Z_0^+$.

Theorem 2. *When $\alpha = \beta$, we have the following identities,*

$$\varphi_{(1)} = \varphi_{(2)} , \qquad \varphi_{(3)} = \varphi_{(4)} , \qquad \varphi_{(5)} = \varphi_{(6)} , \tag{49}$$
$$\varphi_{(7)} = \varphi_{(8)} , \qquad \varphi_{(9)} = \varphi_{(10)} , \quad \text{and} \quad \varphi_{(11)} = \varphi_{(12)} , \tag{50}$$

respectively.

Proof. It is clear, because they overlap each other when $\alpha = \beta$, respectively.

Theorem 3. *We have the following identities,*

$$\varphi_{(1)} = \varphi_{(3)} \quad \textit{for} \quad (\alpha - 1) \in Z_0^+, \tag{51}$$

$$\varphi_{(2)} = \varphi_{(4)} \quad \textit{for} \quad (\beta - 1) \in Z_0^+, \tag{52}$$

$$\varphi_{(5)} = \varphi_{(7)} \quad \textit{for} \quad (\alpha - \gamma) \in Z_0^+, \tag{53}$$

$$\varphi_{(6)} = \varphi_{(8)} \quad \textit{for} \quad (\beta - \gamma) \in Z_0^+, \tag{54}$$

$$\varphi_{(9)} = \varphi_{(11)} \quad \textit{for} \quad (\gamma - \alpha - 1) \in Z_0^+, \tag{55}$$

and

$$\varphi_{(10)} = \varphi_{(12)} \quad \textit{for} \quad (\gamma - \beta - 1) \in Z_0^+, \tag{56}$$

respectively.

Proof. Let $u = u(z) \in \wp^\circ$ and $v = v(z) \in \wp^\circ$, we have then

$$(u \cdot v)_\nu = (v \cdot u)_\nu \quad \text{for} \quad \nu \in Z_0^+, \tag{57}$$

and

$$(u \cdot v)_\nu \neq (v \cdot u)_\nu \quad \text{for} \quad u \neq v, \ \text{and} \ \nu \notin Z_0^+. \tag{58}$$

Therefore, we have this theorem clearly. ([1] Vol.1)

Theorem 4. We have the following identities,

$$\varphi_{(1)} = \varphi_{(2)} = \varphi_{(3)} = \varphi_{(4)} \quad \text{for} \quad \alpha = \beta, \ (\alpha - 1) \in Z_0^+, \tag{59}$$

$$\varphi_{(5)} = \varphi_{(6)} = \varphi_{(7)} = \varphi_{(8)} \quad \text{for} \quad \alpha = \beta, \ (\alpha - \gamma) \in Z_0^+, \tag{60}$$

and

$$\varphi_{(9)} = \varphi_{(10)} = \varphi_{(11)} = \varphi_{(12)} \quad \text{for} \quad \alpha = \beta, \ (\gamma - \alpha - 1) \in Z_0^+. \tag{61}$$

Proof. It is clear by Theorems 2. and 3.

Chapter 2. More familiar forms of the solutions obtained in Chapter 1 and Kummer's twenty-four functions

§1. More familiar forms of the solutions Group I in Chap. 1, § 1.

Theorem 1. *By the fractional calculus of products (using the generalized Leibniz rule) we have* ([6] Vol. 1 & [7])

$$\varphi_{(1)} = K\left(z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1}\right)_{\alpha-1} = z^{1-\gamma}(1-z)^{\gamma-\beta-1}{}_2F_1\left(1+\beta-\gamma, 1-\alpha; 2-\gamma; \frac{z}{z-1}\right) \tag{1}$$

$$= V_{(20)}$$

for $\left|(z^{\alpha-\gamma})_{\alpha-1-n}\right| < \infty$ ($n \in Z^+ \cup \{0\}$), $z \neq 0, 1$ and $|z/(z-1)| < 1$, where ${}_2F_1$ is the usual Hypergeometric functions of Gauss.

Note. For the notations $V_{(k)}$ ($k = 1, 2, \cdots, 24$) refer to the list shown in §2.

Proof. We have

$$\varphi_{(1)} = K\sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha-n)\Gamma(n+1)}\left(z^{\alpha-\gamma}\right)_{\alpha-1-n}\left((z-1)^{\gamma-\beta-1}\right)_n \qquad (z \neq 0, 1) \tag{2}$$

$$= Ke^{-i\pi(\alpha-1)}z^{1-\gamma}(z-1)^{\gamma-\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha)\Gamma(\gamma-1-n)\Gamma(\beta+1-\gamma+n)}{n!\,\Gamma(\alpha-n)\Gamma(\gamma-\alpha)\Gamma(\beta+1-\gamma)}\left(\frac{z}{z-1}\right)^n \tag{3}$$

$$= Ke^{i\pi(\gamma-\alpha-\beta)}\frac{\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)}z^{1-\gamma}(1-z)^{\gamma-\beta-1}\cdot{}_2F_1\left(\beta+1-\gamma, 1-\alpha; 2-\gamma; \frac{z}{z-1}\right) \tag{4}$$

under the conditions, since

$$\Gamma(\alpha - n) = (-1)^{-n}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \qquad (n \in Z^+ \cup \{0\}). \tag{5}$$

Therefore, choosing

$$K = 1/M \qquad \left(M = e^{i\pi(\gamma-\alpha-\beta)}\Gamma(\gamma-1)/\Gamma(\gamma-\alpha)\right) \tag{6}$$

we have (1) from (4).

By the change of α and β in (1), we have

$$z^{1-\gamma}(1-z)^{\gamma-\alpha-1}{}_2F_1\left(1+\alpha-\gamma, 1-\beta; 2-\gamma; \frac{z}{z-1}\right) = V_{(19)}\,. \tag{7}$$

Theorem 2. *By the fractional calculus of products, we have*

$$\varphi_{(3)} = K\left((z-1)^{\gamma-\beta-1}\cdot z^{\alpha-\gamma}\right)_{\alpha-1} = z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(1-\alpha, \gamma-\alpha; 1-\alpha-\beta+\gamma; 1-\frac{1}{z}\right) \tag{8}$$

$$= V_{(23)}$$

for $\left|\left((z-1)^{\gamma-\beta-1}\right)_{\alpha-1-n}\right| < \infty$ $(n \in \mathbb{Z}^+ \cup \{0\})$, $z \neq 0, 1$ *and* $|(z-1)/z| < 1$.

Proof. We have

$$\varphi_{(3)} = K\sum_{n=0}^{\infty}\frac{\Gamma(\alpha)}{\Gamma(\alpha-n)\Gamma(n+1)}\left((z-1)^{\gamma-\beta-1}\right)_{\alpha-1-n}(z^{\alpha-\gamma})_n \qquad (z \neq 0, 1) \tag{9}$$

$$= Ke^{i\pi(\gamma-\beta-1)}z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\alpha)\Gamma(\alpha+\beta-\gamma-n)\Gamma(\gamma-\alpha+n)}{n!\,\Gamma(\alpha-n)\Gamma(1+\beta-\gamma)\Gamma(\gamma-\alpha)}\left(\frac{1-z}{z}\right)^n \tag{10}$$

$$= Ke^{i\pi(\gamma-\beta-1)}\frac{\Gamma(\alpha+\beta-\gamma)}{\Gamma(1+\beta-\gamma)}z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(1-\alpha, \gamma-\alpha; 1-\alpha-\beta+\gamma; 1-\frac{1}{z}\right) \tag{11}$$

under the conditions. Therefore, choosing

$$K = 1/M \qquad \left(M = e^{i\pi(\gamma-\beta-1)}\Gamma(\alpha+\beta-\gamma)/\Gamma(1+\beta-\gamma)\right), \tag{12}$$

we have (8) from (11).

By the change of α and β in (8), we have

$$z^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(1-\beta, \gamma-\beta; 1-\alpha-\beta+\gamma; 1-\frac{1}{z}\right) = V_{(24)}\,. \tag{13}$$

Theorem 3. *Without the use of generalized Leibniz rule, we have*

$$\varphi_{(1)} = K\left(z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\right)_{\alpha-1} = (1-z)^{-\beta}{}_2F_1\left(\gamma-\alpha, \beta; \beta-\alpha+1; \frac{1}{1-z}\right) \tag{14}$$

$$= V_{(15)},$$

for $\left|\left((1-z)^{\alpha-\beta-1-k}\right)_{\alpha-1}\right| < \infty$ $(k \in \mathbb{Z}^+ \cup \{0\})$, $z \neq 0, 1$ *and* $|1-z| > 1$.

Proof. Using the identity

$$z^{\lambda} = (z-1)^{\lambda}\left(1-\frac{1}{1-z}\right)^{\lambda} = (z-1)^{\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}(1-z)^{-k} \qquad (|1-z|>1) \tag{15}$$

we have

$$\varphi_{(1)} = K\left(z^{\lambda}\cdot(z-1)^{\gamma-\beta-1}\right)_{\alpha-1} \qquad (\lambda = \alpha-\gamma) \quad (z \neq 0, 1) \tag{16}$$

$$= K\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}\left((1-z)^{-k}(z-1)^{\lambda+\gamma-\beta-1}\right)_{\alpha-1} \tag{17}$$

$$= Ke^{i\pi(\alpha-\beta-1)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\alpha-\gamma+1)}{k!\,\Gamma(\alpha-\gamma+1-k)}\left((1-z)^{\alpha-\beta-1-k}\right)_{\alpha-1} \tag{18}$$

$$= Ke^{i\pi(\alpha-\beta-1)}(1-z)^{-\beta}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\alpha-\gamma+1)\Gamma(\beta+k)}{k!\,\Gamma(\alpha-\gamma+1-k)\Gamma(-\alpha+\beta+1+k)}(1-z)^{-k} \tag{19}$$

$$= Ke^{i\pi(\alpha-\beta-1)}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(1-z)^{-\beta}{}_2F_1\left(\gamma-\alpha,\beta;\beta-\alpha+1;\frac{1}{1-z}\right), \tag{20}$$

under the conditions.

Therefore choosing

$$K=1/M \qquad \left(M=e^{i\pi(\alpha-\beta-1)}\Gamma(\beta)/\Gamma(\beta-\alpha+1)\right) \tag{21}$$

we have (14) from (20).

By the change α and β in (14), we have

$$(1-z)^{-\alpha}{}_2F_1\left(\gamma-\beta,\alpha;\alpha-\beta+1;\frac{1}{1-z}\right)=V_{(11)}. \tag{22}$$

Theorem 4. *Without the use of generalized Leibniz rule, we have*

$$\varphi_{(1)}=K\left(z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\right)_{\alpha-1}=(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;1-\alpha-\beta+\gamma;1-z) \tag{23}$$
$$=V_{(21)}$$

for $\left|\left((1-z)^{k+\gamma-\beta-1}\right)_{\alpha-1}\right|<\infty \quad (k\in\mathbb{Z}^+\cup\{0\}),\ z\neq 0,1$ *and* $|1-z|<1$.

Proof. Using the identity

$$z^\lambda=(1-(1-z))^\lambda=\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}(1-z)^k \quad (|1-z|<1) \tag{24}$$

we have

$$\varphi_{(1)}=K\left(z^\lambda\cdot(z-1)^{\gamma-\beta-1}\right)_{\alpha-1} \qquad (\lambda=\alpha-\gamma)\quad (z\neq 0,1) \tag{25}$$

$$=K\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}\left((1-z)^k(z-1)^{\gamma-\beta-1}\right)_{\alpha-1} \tag{26}$$

$$=Ke^{i\pi(\gamma-\beta-1)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}\left((1-z)^{k+\gamma-\beta-1}\right)_{\alpha-1} \tag{27}$$

$$=Ke^{i\pi(\gamma-\beta-1)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\alpha-\gamma+1)\Gamma(\alpha+\beta-\gamma-k)}{\Gamma(k+1)\Gamma(\alpha-\gamma+1-k)\Gamma(1+\beta-\gamma-k)}(1-z)^{k+\gamma-\alpha-\beta} \tag{28}$$

$$=Ke^{i\pi(\gamma-\beta-1)}\frac{\Gamma(\alpha+\beta-\gamma)}{\Gamma(1+\beta-\gamma)}(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;1-\alpha-\beta;1-z) \tag{29}$$

under the conditions. Therefore, choosing

$$K=1/M \qquad \left(M=e^{i\pi(\gamma-\beta-1)}\Gamma(\alpha+\beta-\gamma)/\Gamma(1+\beta-\gamma)\right) \tag{30}$$

we have (23) from (29).

By the change α and β in (23) we have (23) itself again.

Theorem 5. *Without the use of generalized Leibniz rule, we have*

$$\varphi_{(1)}=K\left(z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\right)_{\alpha-1}=(-z)^{-\beta}{}_2F_1\left(\beta-\gamma+1,\beta;\beta-\alpha+1;\frac{1}{z}\right) \tag{31}$$
$$=V_{(13)}$$

for $\left|\left(z^{\alpha-\beta-1-k}\right)_{\alpha-1}\right|<\infty \quad (k\in\mathbb{Z}^+\cup\{0\}),\ z\neq 0,1$ *and* $|z|>1$.

Proof. Using the identity

$$(z-1)^\lambda=z^\lambda\left(1-\frac{1}{z}\right)^\lambda=z^\lambda\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}z^{-k} \quad (|z|>1) \tag{32}$$

$$\varphi_{(1)} = K\left(z^{\alpha-\gamma}(z-1)^{\lambda}\right)_{\alpha-1} \qquad (\lambda = \gamma-\beta-1) \quad (z \neq 0,1) \tag{33}$$

$$= K\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\gamma-\beta)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)}\left(z^{\alpha-\beta-1-k}\right)_{\alpha-1} \tag{34}$$

$$= -Ke^{-i\pi\alpha}z^{-\beta}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\gamma-\beta)\Gamma(\beta+k)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)\Gamma(\beta+1+k-\alpha)}z^{-k} \tag{35}$$

$$= -Ke^{i\pi(\beta-\alpha)}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(-z)^{-\beta}{}_2F_1\left(\beta-\gamma+1,\beta;\beta-\alpha+1;\frac{1}{z}\right) \tag{36}$$

under the conditions. Therfore, choosing

$$K = 1/M \qquad \left(M = -e^{i\pi(\beta-\alpha)}\Gamma(\beta)/\Gamma(\beta-\alpha+1)\right) \tag{37}$$

we have (31) from (36).

By the change of α and β in (31), we have

$$(-z)^{-\alpha}{}_2F_1\left(\alpha-\gamma+1,\alpha;\alpha-\beta+1;\frac{1}{z}\right) = V_{(9)}\,. \tag{38}$$

Theorem 6. *Without the use of generalized Leipniz rule, we have*

$$\varphi_{(1)} = K\left(z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\right)_{\alpha-1} = z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z) \tag{39}$$

$$= V_{(17)}\,,$$

for $\left|\left(z^{k+\alpha-\gamma}\right)_{\alpha-1}\right| < \infty \quad (k \in \mathbb{Z}^+ \cup \{0\}),\quad z \neq 0,1$ *and* $|z|<1$.

Proof. Using the identity

$$(z-1)^{\lambda} = e^{i\pi\lambda}(1-z)^{\lambda} = e^{i\pi\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}z^k \qquad (|z|<1) \tag{40}$$

we have

$$\varphi_{(1)} = K\left(z^{\alpha-\gamma}\cdot(z-1)^{\lambda}\right)_{\alpha-1} \qquad (\lambda = \gamma-\beta-1) \quad (z \neq 0,1) \tag{41}$$

$$= Ke^{i\pi(\gamma-\beta-1)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1-k)}\left(z^{k+\alpha-\gamma}\right)_{\alpha-1} \tag{42}$$

$$= Ke^{i\pi(\gamma-\alpha-\beta)}z^{1-\gamma}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\gamma-\beta)\Gamma(\gamma-1-k)}{\Gamma(k+1)\Gamma(\gamma-\beta-k)\Gamma(\gamma-\alpha-k)}z^k \tag{43}$$

$$= Ke^{i\pi(\gamma-\alpha-\beta)}\frac{\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)}z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z) \tag{44}$$

under the conditions. Therefore, choosing

$$K = 1/M, \qquad \left(M = e^{i\pi(\gamma-\alpha-\beta)}\Gamma(\gamma-1)/\Gamma(\gamma-\alpha)\right) \tag{45}$$

we have (39) from (44).

By the change α and β in (39), we have (39) itself again.

§ 2. Commentaries

(I) When none of the numbers γ, $\alpha-\beta$, $\gamma-\alpha-\beta$, is equal to an integer, each of the following twenty-four functions (due to Kummer) satisfies the homogeneous Gauss equation § 1, (0) in Chap. 1. [8].

List of the twenty-four functions by Kummer

$$V_{(1)}={}_2F_1(\alpha,\beta;\gamma;z)$$

$$V_{(2)}=(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z)$$

$$V_{(3)}=(1-z)^{-\alpha}{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right)$$

$$V_{(4)}=(1-z)^{-\beta}{}_2F_1\left(\gamma-\alpha,\beta;\gamma;\frac{z}{z-1}\right)$$

$$V_{(5)}={}_2F_1(\alpha,\beta;\alpha+\beta+1-\gamma;1-z)$$

$$V_{(6)}=z^{1-\gamma}{}_2F_1(\alpha+1-\gamma,\beta+1-\gamma;\alpha+\beta+1-\gamma;1$$

$$V_{(7)}=z^{-\alpha}{}_2F_1\left(\alpha,\alpha+1-\gamma;\alpha+\beta+1-\gamma;1-\frac{1}{z}\right)$$

$$V_{(8)}=z^{-\beta}{}_2F_1\left(\beta+1-\gamma,\beta;\alpha+\beta+1-\gamma;1-\frac{1}{z}\right)$$

$$V_{(9)}=(-z)^{-\alpha}{}_2F_1\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}\right)$$

$$V_{(10)}=(-z)^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(1-\beta,\gamma-\beta;\alpha+1-\beta;\frac{1}{z}\right)$$

$$V_{(11)}=(1-z)^{-\alpha}{}_2F_1\left(\alpha,\gamma-\beta;\alpha+1-\beta;\frac{1}{1-z}\right)$$

$$V_{(12)}=(-z)^{1-\gamma}(1-z)^{\gamma-\alpha-1}{}_2F_1\left(\alpha+1-\gamma,1-\beta;\alpha+1-\beta;\frac{1}{1-z}\right)$$

$$V_{(13)}=(-z)^{-\beta}{}_2F_1\left(\beta+1-\gamma,\beta;\beta+1-\alpha;\frac{1}{z}\right)$$

$$V_{(14)}=(-z)^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(1-\alpha,\gamma-\alpha;\beta+1-\alpha;\frac{1}{z}\right)$$

$$V_{(15)}=(1-z)^{-\beta}{}_2F_1\left(\beta,\gamma-\alpha;\beta+1-\alpha;\frac{1}{1-z}\right)$$

$$V_{(16)}=(-z)^{1-\gamma}(1-z)^{\gamma-\beta-1}{}_2F_1\left(\beta+1-\gamma,1-\alpha;\beta+1-\alpha;\frac{1}{1-z}\right)$$

$$V_{(17)}=z^{1-\gamma}{}_2F_1(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;z)$$

$$V_{(18)}=z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1(1-\alpha,1-\beta;2-\gamma;z)$$

$$V_{(19)}=z^{1-\gamma}(1-z)^{\gamma-\alpha-1}{}_2F_1\left(\alpha+1-\gamma,1-\beta;2-\gamma;\frac{z}{z-1}\right)$$

$$V_{(20)}=z^{1-\gamma}(1-z)^{\gamma-\beta-1}{}_2F_1\left(\beta+1-\gamma,1-\alpha;2-\gamma;\frac{z}{z-1}\right)$$

$$V_{(21)}=(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1-\alpha-\beta;1-z)$$

$$V_{(22)}=z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1(1-\alpha,1-\beta;\gamma+1-\alpha-\beta;1-z)$$

$$V_{(23)}=z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(\gamma-\alpha,1-\alpha;\gamma+1-\alpha-\beta;1-\frac{1}{z}\right)$$

$$V_{(24)}=z^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1\left(\gamma-\beta,1-\beta;\gamma+1-\alpha-\beta;1-\frac{1}{z}\right).$$

Moreover, we have the following six identities.

(i) $V_{(1)}=V_{(2)}=V_{(3)}=V_{(4)}$,

(ii) $V_{(5)}=V_{(6)}=V_{(7)}=V_{(8)}$,

(iii) $V_{(9)}=V_{(10)}=V_{(11)}=V_{(12)}$,

(iv) $V_{(13)}=V_{(14)}=V_{(15)}=V_{(16)}$,

(v) $V_{(17)}=V_{(18)}=V_{(19)}=V_{(20)}$,

(vi) $V_{(21)}=V_{(22)}=V_{(23)}=V_{(24)}$.

(II) By our N-fractional calculus operator N^ν method to the homogeneous Gauss equation we obtained the solutions shown in Chap. 1, which have the fractional differintegrated forms respectively.

The translations from our solutions Chap.1, Group I to the more familiar forms which contain the well known Gauss Hypergeometric functions yield, as we see in Chap. 2, § 1,

$$V_{(20)},\ V_{(19)},\ V_{(23)},\ V_{(24)},\ V_{(15)},\ V_{(11)},\ V_{(21)},\ V_{(13)},\ V_{(9)},\ V_{(17)}$$

(refer to the list described above).

In the same way as the procedure shown in Chapter 2. § 1, the transrations from the solutions of Group II yield

$$V_{(3)},\ V_{(4)},\ V_{(24)},\ V_{(23)},\ V_{(12)},\ V_{(16)},\ V_{(22)},\ V_{(9)},\ V_{(13)},\ V_{(1)},$$

and the transrations from the solutions of Group III yield

$$V_{(19)},\ V_{(20)},\ V_{(7)},\ V_{(8)},\ V_{(11)},\ V_{(15)},\ V_{(5)},\ V_{(10)},\ V_{(14)},\ V_{(18)}\ .$$

Therefore, we see that almost all functions of the group $V_{(1)} \longleftrightarrow V_{(24)}$ in the list described above can be derived directly from our solutions of the Groups I, II and III in Chap. 1, which have fractional differintegrated forms, except only two functions $V_{(2)}$ and $V_{(6)}$. (For the calculations from the solutions of Group II and III, refer to J F C Vol. 10, November 1996, pp 9 - 23.)

However, we have the relationships (i) and (ii) respectively.

Therefore, we can derive the Kummer's twenty-four functions from our solutions in Chap. 1.

That is, the solutions obtained by our N^ν operator method cover the Kummer's 24 functions.

(III) All mathematicians should compare our N-fractional calculus, N-transformation and N-fractional calculus operator N^νmethod to the ordinary and partial differential equations with that of other fractional calculus, for example, Riemann-Liouville, Weyl, Osler, Oldham and Spanier's ones.

(IV) Hitherto, only the solutions of the forms in Chap. 1, Group I had been treated in the applications of fractional calculus to differential equations. However, this is insufficient. Namely, we must add the solutions such as the forms of Group II and III, which are shown in Chap. 1, § 1. For the solutions to the other differential equations the situations are same.

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; N- method to nearly simple harmonic vibration equations, J. Frac. Calc. Vol. 15, May (1999), 67 - 72.

[6] S.- T. Tu and D.- K. Chyan ; A certain family of infinite series, differintegrable functions and Psi functions, J. Frac. Calc. Vol. 7, May (1995), 41 - 46.

[7] K. Nishimoto ; N- method to fractional differintegral equations $\varphi_{\pm m/n} + \varphi \cdot a = f\ (a \neq 0,\ m < n.\ m, n \in Z^+)$, J. Frac. Calc. Vol. 14, Nov. (1998), 41 - 44.

[8] T. Miyakoda and K. Nishimoto ; N- method to fractional differintegral equations $\varphi_{\pm 3/4} + \varphi \cdot a = f \ (a \neq 0)$, J. Frac. Calc. Vol. 15, May (1999), 7 - 12.

[9] K. Nishimoto ; N- Fractional Calculus of the Power and Logarithmic Functions, and some Identities, J. Frac. Calc. Vol. 21, May (2002), 1 - 6.

[10] K. Nishimoto ; Some Theorems for N- Fractional Calculus of Logarithmic Functions I, J. Frac. Calc. Vol. 21, May (2002), 7 - 12.

[11] K. Nishimoto ; Solutions of Gauss equation in fractional calculus, J. Frac. Calc. Vol. 3, May (1993), 29 - 37.

[12] K. Nishimoto ; N- method to Weber equations, J. Frac. Calc. Vol. 14, Nov. (1998), 1 - 8.

[13] K. Nishimoto ; N- method to generalized Laguerre equations, J. Frac.Calc.Vol.14, Nov., (1998), 9 - 21.

[14] K. Nishimoto and Susana S. de Romero ; N- fractional calculus operator N^{ν} method to nonhomogeneous and homogeneous Whittaker equations (I), J. Frac. Calc.Vol.9, May (1996), 17 - 22.

[15] Susana S. de Romero and K. Nishimoto ; N- fractional calculus operator N^{ν} method to nonhomogeneous and homogeneous Whittaker equations (I I) (Some illustrative examples), J. Frac. Calc.Vol.12, Nov., (1997), 29 - 35.

[16] K. Nishimoto ; Operator N^{ν} method to a generalized linear second order homogeneous ordinary differential equations J. Frac. Calc.Vol.11, May, (1997), 1 - 9.

[17] David Dummit and Richard M. Foote ; Abstract Algebra, Prentice Hall (1991).

[18] K. B. Oldham and J. Spanier ; The Fractional Calculus, Academic Press (1974).

[19] A.C. McBride ; Fractional Calculus and Integral Transforms of Generalized Functions, Research Notes, Vol. 31, (1979), Pitman.

[20] S.G. Samko, A.A. Kilbas and O.I. Marichev ; Fractional Integrals and Derivatives, and Some Their Applications (1987), Nauka, USSR.

[21] K. S. Miller and B. Ross ; An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, (1993).

[22] V. Kiryakova ; Generalized fractional calculus and applications, Pitman Research Notes, No.301, (1994), Longman.

[23] A. Carpinteri and F. Mainardi (Ed.); Fractals and Fractional Calculus in Continuum Mechanics (1997), Springer, Wien, New York.

[24] Igor Podlubny ; Fractional Differential Equations (1999), Academic Press.

[25] R. Hilfer (Ed.) ; Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapore, New Jersey, London, Hong Kong.